

Finite versus infinite: an insufficient shift

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The shift graph \mathcal{G}_S is defined on the space of infinite subsets of natural numbers by letting two sets be adjacent if one can be obtained from the other by removing its least element. We show that this graph is not a minimum among the graphs of the form \mathcal{G}_f defined on some Polish space X , where two distinct points are adjacent if one can be obtained from the other by a given Borel function $f : X \rightarrow X$. This answers the primary outstanding question from [KST99].

A *directed graph* is a pair $\mathcal{G} = (X, R)$ where R is an irreflexive binary relation on X . A homomorphism from $\mathcal{G} = (X, R)$ to $\mathcal{G}' = (X', R')$ is a map $h : X \rightarrow X'$ such that $(x, y) \in R$ implies $(h(x), h(y)) \in R'$ for all $x, y \in X$. A *coloring* of \mathcal{G} is a map $c : X \rightarrow Y$ such that $(x_1, x_2) \in R$ implies $c(x_1) \neq c(x_2)$ for all $(x_1, x_2) \in X \times X$. In case X is a topological space, the *Borel chromatic number* $\chi_B(\mathcal{G})$ of \mathcal{G} is defined by

$$\chi_B(\mathcal{G}) = \min\{|c(X)| \mid c : X \rightarrow Y \text{ is a Borel coloring of } \mathcal{G} \text{ in a Polish space } Y\},$$

where $|c(X)|$ denotes the cardinality of the range of c .

In this note we only deal with graphs generated by a function. Let X be a Polish space and $f : X \rightarrow X$ is a Borel map. We let $\mathcal{D}_f = (X, D_f)$ be the directed graph given by

$$x D_f y \iff x \neq y \wedge f(x) = y.$$

We also consider its symmetric counterpart $\mathcal{G}_f = (X, R_f)$ given by

$$x R_f y \iff x \neq y \wedge (f(x) = y \vee f(y) = x).$$

Notice that clearly $\chi_B(\mathcal{G}_f) = \chi_B(\mathcal{D}_f)$.

The following example has drawn considerable attention in the study of Borel chromatic number [CM14, DPT06, DPT12, DPT15]. Let $X = [\omega]^\omega$ be the set of infinite sets of natural numbers with topology induced from the Cantor space 2^ω when $Y \subseteq \omega$ is identified with its characteristic function $\chi_Y : \omega \rightarrow 2$. The shift operation $S : [\omega]^\omega \rightarrow$

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$[\omega]^\omega$ is the continuous map defined by $S(Y) = Y \setminus \{\min Y\}$. While \mathcal{G}_S is an acyclic graph and has therefore chromatic number 2, it follows from the Galvin–Prikrý Theorem [GP73] that $\chi_B(\mathcal{G}_S) = \aleph_0$.

Kechris, Solecki and Todorćević [KST99, Problem 8.1] (see also [DPT15, Section 3] and [Mil08]) asked whether the following is true: If X is a Polish space and $f : X \rightarrow X$ is a Borel \aleph_0 -to-1 function, then exactly one of the following holds:

1. The Borel chromatic number of \mathcal{G}_f is finite;
2. There is a continuous homomorphism from \mathcal{G}_S to \mathcal{G}_f .

We show that the answer is negative, namely:

Theorem 1. *There exists a Polish space X together with a continuous \aleph_0 -to-1 function $f : X \rightarrow X$ such that $\chi_B(\mathcal{G}_f) = \aleph_0$ and there is no Borel homomorphism from \mathcal{G}_S to \mathcal{G}_f .*

We do not have any explicit example witnessing the above existential statement. This is because our proof consists of showing that a certain subset of the set of graphs with the above property is a true Π_2^1 set in some suitable standard Borel space.

We can however be a bit more specific. If P is a binary relation on ω , let \vec{P} be the closed subset of $[\omega]^\omega$ defined by

$$\vec{P} = \{(n_i)_{i \in \omega} \in [\omega]^\omega \mid \forall i \in \omega \ n_i P n_{i+1}\},$$

where an element of $[\omega]^\omega$ is identified with the enumeration $(n_i)_{i \in \omega}$ of its elements in strictly increasing order. If $\mathcal{G} = (X, R)$ is a directed graph and $Y \subseteq X$ let us denote by $\mathcal{G}|Y$ the restriction of \mathcal{G} to Y given by $(Y, R \cap (Y \times Y))$.

The proof of Theorem 1 actually yields the following result.

Scholium 2. *There exists a binary relation P on ω such that $\chi_B(\mathcal{G}_S|\vec{P}) = \aleph_0$ and there is no Borel homomorphism from \mathcal{G}_S to $\mathcal{G}_S|\vec{P}$.*

Before proving Theorem 1 we want to recall a definition and establish a simple but important lemma. A binary relation $P \subseteq A \times A$ on some set A is called a *better-binary-relation* if for all continuous map $\varphi : [\omega]^\omega \rightarrow A$, where A is considered a discrete space, there exists $X \in [\omega]^\omega$ such that $(\varphi(X), \varphi(S(X))) \in P$, or in fewer words, if there is no continuous homomorphism from \mathcal{D}_S to (A, P^c) , where $P^c = (A \times A) \setminus P$. This notion which first appeared in [She82] is a straightforward generalization to arbitrary binary relations of that of *better-quasi-order*¹ due to Nash-Williams [NW65]. For more on better-quasi-orders we refer the reader to [Sim85, Mar94, CP14] and to the author’s PhD thesis [Peq15].

Lemma 3. *Let $P \subseteq \omega \times \omega$ be an irreflexive binary relation on ω . Then the following are equivalent:*

1. P^c is not a better-binary-relation,

¹A *better-quasi-order* is just a transitive better-binary-relation, as a better-binary-relation is necessarily reflexive.

2. there exists a continuous homomorphism from \mathcal{D}_S to (ω, P) ,
3. there exists a continuous homomorphism from \mathcal{D}_S to $\mathcal{D}_S|\vec{P}$.
4. there exists a Borel homomorphism from \mathcal{G}_S to $\mathcal{G}_S|\vec{P}$.

Proof. (1) \leftrightarrow (2) follows from the definition of a better-binary-relation.

(2) \rightarrow (3): Assume that $\varphi : [\omega]^\omega \rightarrow \omega$ is a continuous homomorphism from \mathcal{D}_S to (ω, P) . Since the usual order on ω is a better-quasi-order, by applying the Galvin–Prikry theorem to the Borel partition:

$$[\omega]^\omega = \{X \mid \Phi(X) \leq \Phi(S(X))\} \cup \{X \mid \Phi(X) > \Phi(S(X))\},$$

and eventually restricting φ to $[Y]^\omega$ for some $Y \in [\omega]^\omega$, we can assume without loss of generality that $\varphi(X) \leq \varphi(S(X))$ for every $X \in [\omega]^\omega$. As P is irreflexive and φ is a homomorphism, we actually have $\varphi(X) < \varphi(S(X))$ for every $X \in [\omega]^\omega$. We define $\Phi : [\omega]^\omega \rightarrow \vec{P}$ by setting $\Phi(X) = \{\varphi(S^n(X)) \mid n \in \omega\}$ for every $X \in [\omega]^\omega$. Clearly Φ is a well defined continuous homomorphism from \mathcal{D}_S to $\mathcal{D}_S|\vec{P}$ as desired.

(3) \rightarrow (4) is obvious.

(4) \rightarrow (2): Suppose that Φ is a Borel homomorphism from \mathcal{G}_S to $\mathcal{G}_S|\vec{P}$. Applying the Galvin–Prikry theorem to the Borel partition

$$[\omega]^\omega = \{X \mid S(\Phi(X)) = \Phi(S(X))\} \cup \{X \mid S(\Phi(S(X))) = \Phi(X)\},$$

and eventually restricting Φ to $[Y]^\omega$ for some $Y \in [\omega]^\omega$, we can suppose without loss of generality that Φ is actually a homomorphism from \mathcal{D}_S to $\mathcal{D}_S|\vec{P}$. By eventually restricting further Φ to $[Z]^\omega$ for some $Z \in [\omega]^\omega$, we can assume that Φ is continuous ([Mat77, section 6], [Sim85, Theorem 3.5] and [PV92, Proposition 3.2]). We then define $\varphi : [\omega]^\omega \rightarrow \omega$ by $\varphi(X) = \min \Phi(X)$ for all $X \in [\omega]^\omega$. Since Φ is a homomorphism from \mathcal{D}_S to \mathcal{D}_S , we have $\varphi(S(X)) = \min \Phi(S(X)) = \min S(\Phi(X))$, and as $\Phi(X) \in \vec{P}$ it follows that $\varphi(X) P \varphi(S(X))$ for all $X \in [\omega]^\omega$. Hence φ is a homomorphism from \mathcal{D}_S to (ω, P) and clearly φ is continuous, as desired. \square

Proof of Theorem 1. We confine ourselves to the graphs $\mathcal{G}_S|C$ obtained by restricting \mathcal{G}_S to some closed subset C of $[\omega]^\omega$ closed under the shift operation, i.e. such that $S(X) \in C$ for all $X \in C$. First some notation. Let $[\omega]^{<\omega}$ be the set of finite sets of natural numbers. For $s \in [\omega]^{<\omega}$ and $t \subseteq \omega$ let $s \sqsubseteq t$ denote that s is an *initial segment* of t with respect to the usual order on ω , namely $s \sqsubseteq t$ if and only if $s = t$ or $\exists k \in t$ such that $s = \{n \in t \mid n < k\}$.

We consider the Effros Borel space $\mathcal{F}([\omega]^\omega)$ of closed subsets of $[\omega]^\omega$ (see [Kec95, 12.C]). The σ -algebra of Borel sets of $\mathcal{F}([\omega]^\omega)$ is generated by the sets of the form

$$\{F \in \mathcal{F}([\omega]^\omega) \mid F \cap [s] \neq \emptyset\}$$

where $[s] = \{X \in [\omega]^\omega \mid s \sqsubseteq X\}$, for $s \in [\omega]^{<\omega}$. We identify every closed subset F of $[\omega]^\omega$ with the pruned tree $T_F = \{t \in [\omega]^{<\omega} \mid \exists X \in F \ t \sqsubseteq X\}$ that we view as an element of the product space $2^{[\omega]^{<\omega}}$, with $[\omega]^{<\omega}$ discrete. Notice that the Effros Borel structure

on $\mathcal{F}([\omega]^\omega)$ coincides with the Borel structure induced by $2^{[\omega]^{<\omega}}$ via this identification. Let \mathcal{T} be the set of closed sets $F \in \mathcal{F}([\omega]^\omega)$ such that $S(X) \in F$ for all $X \in F$. Clearly \mathcal{T} is Borel in $\mathcal{F}([\omega]^\omega)$ since

$$F \in \mathcal{T} \iff \forall t \in [\omega]^{<\omega} \forall n [(n < \min t \wedge \{n\} \cup t \in T_F) \rightarrow t \in T_F].$$

We henceforth work within the standard Borel space \mathcal{T} ([Kec95, (13.4)]). The two subsets of \mathcal{T} that we are interested in are the following:

$$\begin{aligned} \mathcal{N} &= \{F \in \mathcal{T} \mid \text{there is no Borel homomorphism from } \mathcal{G}_S \text{ to } \mathcal{G}_S|F\}, \\ \mathcal{F} &= \{F \in \mathcal{T} \mid \chi_B(\mathcal{G}_S|F) < \aleph_0\}. \end{aligned}$$

Clearly $\mathcal{F} \subseteq \mathcal{N}$, as the composition of a coloring with a homomorphism is again a coloring and $\chi_B(\mathcal{G}_S) = \aleph_0$. First we observe that \mathcal{F} is a Σ_2^1 set in \mathcal{T} . While this can be seen by a direct Tarski–Kuratowski computation on the definition of \mathcal{F} , we find it easier to use the neat characterization given by Miller [Mil08, Thm 2.1] which gives: $F \in \mathcal{F}$ if and only if there exists a Borel set $B \subseteq F$ such that for all $X \in F$ there exist $m, n \in \omega$ such that $S^m(X) \in B$ and $S^n(X) \notin B$. We fix a coding of Borel subsets of the Polish space $[\omega]^\omega$, see [Kec95, (35.B)]: let D be a Π_1^1 subset of ω^ω and W be a Δ_1^1 subset of $D \times [\omega]^\omega$ such that $\{W_d \mid d \in D\}$, where $W_d = \{X \mid (d, X) \in W\}$, is the set of all Borel subsets $[\omega]^\omega$. We get

$$\begin{aligned} F \in \mathcal{F} \iff \exists d \in \omega^\omega \Big(d \in D \\ \wedge \forall X \in [\omega]^\omega \Big[X \in F \rightarrow (\exists n S^n(X) \in W_d \wedge \exists n S^n(X) \notin W_d) \Big] \Big). \end{aligned}$$

This clearly gives a Σ_2^1 definition of \mathcal{F} in \mathcal{T} .

Next we show that the inclusion $\mathcal{F} \subsetneq \mathcal{N}$ is strict as witnessed by a closed set of the form $\vec{P}^{\vec{c}}$ for some reflexive binary relation P on ω . This will prove Scholium 2 and finish the proof.

To achieve this, we rely on a deep result due to Marcone [Mar94, Mar95] that we now recall. The set $\mathcal{B} = \{Q \subseteq \omega \times \omega \mid Q \text{ is a better-binary-relation}\}$ is a Π_2^1 -complete subset of the compact Polish space $\mathcal{R} = \{P \subseteq \omega \times \omega \mid P \text{ is reflexive}\}$, where \mathcal{R} is endowed with the topology induced by the product space $2^{\omega \times \omega}$, so in particular \mathcal{B} is not Σ_2^1 in \mathcal{R} . For every reflexive $P \subseteq \omega \times \omega$ let

$$f(P) = \{(n_i)_{i \in \omega} \in [\omega]^\omega \mid \forall i \neg n_i P n_{i+1}\} = \vec{P}^{\vec{c}}.$$

This defines a Δ_2^1 -measurable reduction $f : \mathcal{R} \rightarrow \mathcal{T}$ from \mathcal{B} to \mathcal{N} . To see that $f^{-1}(\mathcal{N}) = \mathcal{B}$, let $P \subseteq \omega \times \omega$ be a reflexive relation. Then $P^{\vec{c}}$ is an irreflexive relation and by Lemma 3, P is a better-binary-relation if and only if $f(P) = \vec{P}^{\vec{c}} \in \mathcal{N}$.

To see that f is Δ_2^1 -measurable notice that for every basic open set $[s] = \{X \in [\omega]^\omega \mid s \subseteq X\}$ of $[\omega]^\omega$ we have

$$f(P) \cap [s] \neq \emptyset \iff \exists (n_i)_{i \in \omega} \in [\omega]^\omega (\forall i \neg n_i P n_{i+1} \wedge s \subseteq (n_i)_{i \in \omega}).$$

It follows that for every Borel subset C of \mathcal{T} the set $f^{-1}(C)$ belongs to the σ -algebra generated by Σ_1^1 , and *a fortiori* to Δ_2^1 .

Suppose towards a contradiction that $f(P) \in \mathcal{N}$ implies $f(P) \in \mathcal{F}$ for all $P \in \mathcal{R}$. Then it follows that $f^{-1}(\mathcal{F}) = \mathcal{B}$. But since \mathcal{F} is Σ_2^1 and Σ_2^1 is closed under preimages by Δ_2^1 -measurable functions [Kec95, (37.3)], we get that \mathcal{B} is Σ_2^1 contradicting Marcone's Theorem. Therefore there exists some $P \in \mathcal{R}$ such that $f(P) \in \mathcal{N} \setminus \mathcal{F}$ as desired. \square

It would be very interesting to find an explicit example of a graph \mathcal{G}_f whose existence is guaranteed by Theorem 1. Notice that by a direct application of a result due to Pouzet [Pou93, Theorem 7] (see also [Mar94, Theorem 1.8]), the binary relation P in Scholium 2 can be chosen such that $P^c = (\omega \times \omega) \setminus P$ is actually a quasi-order, and therefore a better-quasi-order by Lemma 3. We end this note by giving an example of a graph \mathcal{G}_f generated in a similar fashion from a countable better-quasi-order and for which we do not know the Borel chromatic number.

Consider the set $2^{<\omega}$ of finite binary words equipped with the subword ordering, i.e.

$$u \leq v \iff \begin{array}{l} \text{there exists a strictly increasing map } h : |u| \rightarrow |v| \\ \text{such that for every } i < |u| \text{ we have } u(i) = v(h(i)), \end{array}$$

where $|u|$ denotes the length of $u \in 2^{<\omega}$. Consider the closed subspace

$$X = \{(u_n)_{n \in \omega} \in (2^{<\omega})^\omega \mid \forall n \ u_n \not\leq u_{n+1}\}.$$

of the product space $(2^{<\omega})^\omega$, where $2^{<\omega}$ is discrete, and define the corresponding shift operation $S' : X \rightarrow X$, $(u_n)_{n \in \omega} \mapsto (u_{n+1})_{n \in \omega}$. Since $(2^{<\omega}, \leq)$ is a better-quasi-order, there is no Borel homomorphism from \mathcal{G}_S to $\mathcal{G}_{S'}$ as the proof of Lemma 3 also shows. We however ask the following:

Question. What is the Borel chromatic number of $\mathcal{G}_{S'}$?

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